Poll locked. Responses not accepted.

Suppose $f(x, y)$ is a nice, differentiable function that we are trying to optimize subject to the constraint $x^2 + y^2 - 1 = 0$. If we try to solve the Lagrange multipliers equations, which of the following statements is accurate?

- It is possible to find no solutions.
- We are guaranteed to find at least one solution.
- We are guaranteed to find at least two solutions.
- None of the above

Total Results: 13

Example: $f(x,y) = x+y$ $\quad g(x,y) = x^2+y^2-1$

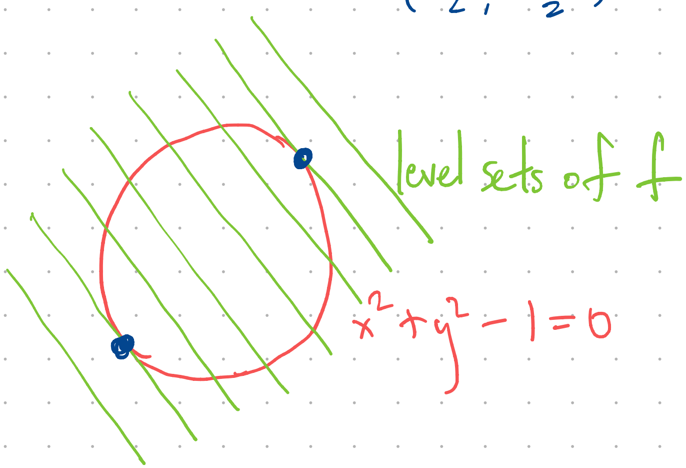
$\nabla f = \langle 1, 1 \rangle \qquad \nabla g = \langle 2x, 2y \rangle$

$$\begin{cases} 1 = 2x\lambda \\ 1 = 2y\lambda \\ x^2+y^2-1 = 0 \end{cases}$$

Get $x=y$ from first two eqs...

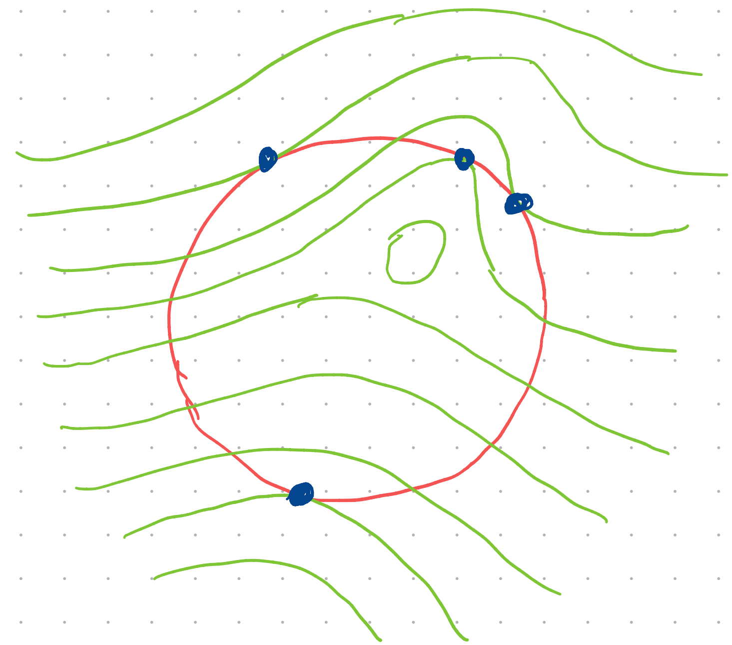
Then solve to find (x,y)

$$= \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \text{ or } \left(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$$



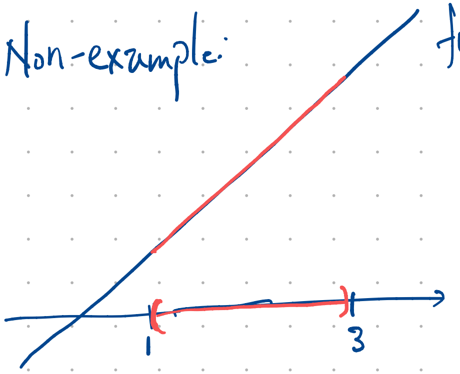
If $f(x,y) = x+y$ then we find two solutions.

What can you say about the # of solutions in general?



Key point: EXTREME VALUE THEOREM

Non-example:



$f(x) = x$

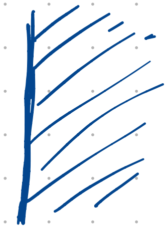
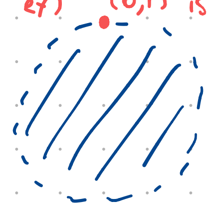
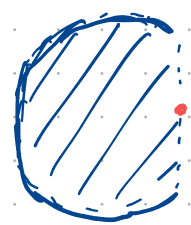
has no max or min on open interval $1 < x < 3$

EVT applies when you have a continuous function defined on a region that is both

(1) closed: Any point that is "infinitesimally close" to the region is actually *in* the region.

(2) bounded: the region does not go "off to infinity".

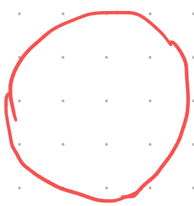
then the function attains an absolute max and min on the region.

region	closed	bounded
 $x \geq 0$	✓	✗
ex) $(0,1)$ is not in region  $x^2+y^2 < 1$	✗	✓
 $x^2+y^2 \leq 1$ & $x < 1/2$	✗	✓

Guideline (true in 99% of situations)

If your region can be algebraically expressed using only $=$ $\leq$ and $\geq$, then it is closed. If you need $<$ or $>$, then it probably isn't closed.

Returning to the problem:

 is closed & bounded

$f(x,y)$ is assumed differentiable, so in particular it is also continuous

EVT $\Longrightarrow$ f has a max and min on the circle, which Lagrange multi would find.

Note: if f attains max and min @ same point, then f must be constant on the constraint region, so there would be ∞ many solutions. (In any case, there are at least 2).

⚠ If constraint region is not closed and bounded, then this logic doesn't apply.

e.g. $g(x,y) = x - y$, $f(x,y) = x + y$

Situation: Optimize $f(x,y,z)$

subject to $g(x,y,z)=0$ and
$h(x,y,z)=0$.

Method outlined in book:

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h & (*) \\ g = 0 \\ h = 0 \end{cases}$$

This is 5 equations in the 5

unknowns $\underbrace{x, y, z,}_{\text{really only care about these @ the end}}$ λ, μ

Alternative method:

Replace $(*)$ with

$$\nabla f \cdot (\nabla g \times \nabla h) = 0$$

i.e.
$$\begin{vmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{vmatrix} = 0 \qquad (**)$$

Advantage: You get 3 eqns in 3 unknowns

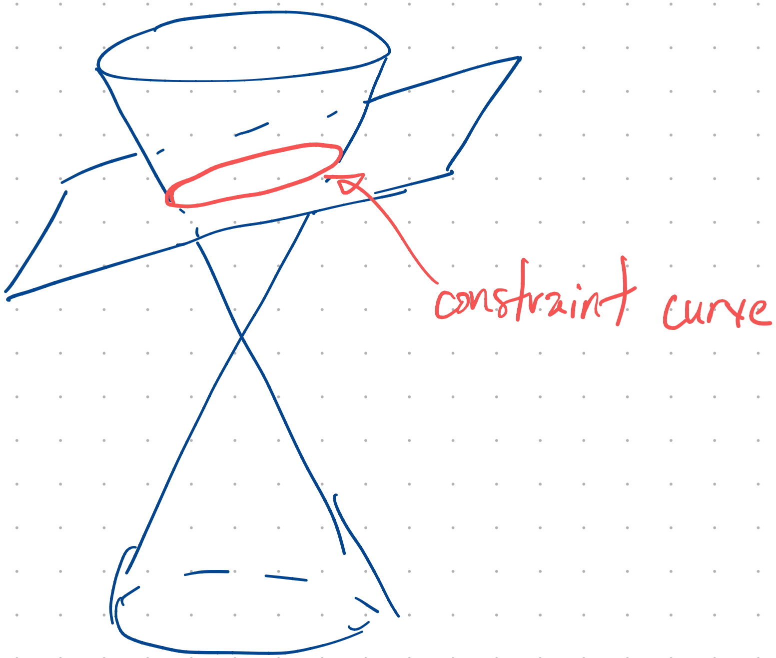
x, y, z

Disadvantage: $(**)$ could be pretty complicated.

ex) $f(x,y,z) = z$

$g(x,y,z) = 4x - 3y + 8z - 5$

$h(x,y,z) = z^2 - x^2 - y^2$



Alternative:

$$\det \begin{bmatrix} 0 & 0 & 1 \\ 4 & -3 & 8 \\ -2x & -2y & 2z \end{bmatrix} = 0$$

the determinant above.

$$\begin{cases} -8y - 6x = 0 \\ 4x - 3y + 8z - 5 \\ z^2 - x^2 - y^2 = 0 \end{cases}$$

Usual method:

$$\begin{cases} 0 = 4\lambda - 2x\mu \\ 0 = -3\lambda - 2y\mu \\ 1 = 8\lambda + 2z\mu \\ 4x - 3y + 8z - 5 = 0 \\ z^2 - x^2 - y^2 = 0 \end{cases}$$